THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2021-22
Lecture 4 practice problems solution
16th February 2022

- The practice problems are meant as exercise to the students. You are **NOT** required to submit your solutions, but you are encouraged to work through all of them in order to understand the course materials. The problems will be uploaded on Fridays and solutions will be uploaded on Wednesdays before the next lecture.
- Please send an email to **echlam@math.cuhk.edu.hk** if you have any questions.

1. Consider $T(z) = \frac{az+b}{cz+d}$ and $S(z) = \frac{ez+f}{gz+h}$, their composition is given by

$$\begin{aligned} T \circ S(z) = T(S(z)) &= \frac{aS(z)+b}{cS(z)+d} \\ &= \frac{a\frac{ez+f}{gz+h}+b}{c\frac{ez+f}{gz+h}+d} \\ &= \frac{(aez+af)+(bgz+bh)}{gz+h} \Big/ \frac{(cez+cf)+(dgz+dh)}{gz+h} \\ &= \frac{(ae+bg)z+(af+bh)}{(ce+dg)z+(cf+dh)} \end{aligned}$$

   Here we can divide cancel out $gz+h$ because it is not identically 0, as S would not be well defined in that case.

   On the other hand, $A_T \cdot A_S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$, so one can see that it corresponds to $T \circ S$.

2. Suppose $f(z) = \frac{az+b}{cz+d}$ fixes 0 and ∞, then $f(0) = \frac{b}{d} = 0$ implies that $b = 0$. And $f(\infty) = \frac{a}{c} = \infty$ implies that $\frac{c}{a} = 0$ so $c = 0$. So $f(z)$ takes the form $\frac{az+0}{0z+d} = \frac{a}{d}z$, where $\frac{a}{d} \neq 0$ since if a is 0, $f \cong 0$ is not a well-defined Möbius transformation.

3. (a) One just has to solve for $T(z) = z$. $\frac{z-i}{z+i} = z$ gives $z^2+(i-1)z+i = 0$. One can solve this using quadratic formula, the root(s) are given by $\frac{1}{2}(1-i \pm \sqrt{(i-1)^2-4i}) = \frac{1}{2}(1-i\pm\sqrt{-6i})$. Here square root means either one of the square roots of a complex number, as explained before. Here I can take $\sqrt{-6i} = \sqrt{3}(1-i)$. So the fixed points are $\frac{1}{2}(1\pm\sqrt{3})(1-i)$. As for ∞, $f(\infty) = \frac{a}{c} = 1$ so it is not a fixed point.

   (b) Solving $z = 3iz$ is simpler because this one is only linear. The fixed point in the plane $\mathbb{C}$ is given by $z = 0$. As for ∞, $T(\infty) = \frac{a}{c} = \frac{1}{0} = \infty$ in the arithmetic of the extended complex plane, so ∞ is a fixed point.

   (c) Again $z = \frac{2iz+1}{2i+1}$ is also linear. $(2i+1)z = 2iz+1$ implies that $z = 1$ is a fixed point. As for ∞, $T(\infty) = \frac{a}{c} = \frac{2i}{0} = \infty$, so again ∞ is a fixed point.